J. Fluid Mech. (1965), *vol.* 22, *part* 4, *pp.* 823–827
Printed in Great Britain

A note on the interaction between surface waves and wind profiles

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(Received 16 November 1964)

The modification of a turbulent wind profile owing to momentum transfer from wind to surface waves is calculated in terms of the power spectrum of the surface-wave slope. The reduction in profile curvature, and hence of the wind-to-wave energy-transfer coefficient, is estimated on the basis of a Neumann spectrum. It appears that this reduction is likely to be small for typical wind speeds.

1. Introduction

Stewart (1961) has pointed out that the transfer of momentum to surface waves from wind blowing over water must affect the wind profile. Invoking basic similarity arguments, he inferred that the mean shear in a turbulent wind would be reduced (relative to its value for flow over a plane wall) by this momentum transfer, but he did not attempt any quantitative calculations. Another, possibly important effect (which Stewart did not mention) of this momentum transfer is the consequent reduction of profile curvature, and hence of the Reynolds stress that is responsible for the momentum transfer (Miles 1957, 1960).

We present here a calculation of the reduction of profile curvature in terms of the power spectrum of the surface-wave slope. It appears, on the assumption of empirical results for both the power spectrum and the mean-square slope [(3.6) and (1.6) below], that this reduction, and hence also the reduction of profile slope, is small for typical wind speeds. (However, the effect of the surface waves on the profile itself remains substantial insofar as they determine the effective roughness.)

Before entering into the details of our calculation, we note that Stewart based his numerical estimates of the effective shear stress associated with the momentum transfer to surface waves, say τ_w, on a series of measurements of wave height and period against wind duration, with wind speed as a parameter. An alternative estimate can be inferred from the empirical result [Munk 1955, using data based on Van Dorn's 1953 measurements]

$$\tau_w = 0{\cdot}68 \times 10^{-6} \rho (g\nu_w)^{-\frac{1}{3}} U_a^3, \tag{1.1}$$

where ρ denotes the density of the air, g the acceleration of gravity, ν_w the kinematic viscosity of the water, and U_a the wind speed at an anemometer height of

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10 m above the surface. Expressing the total shear stress, say τ, in terms of a drag coefficient c_d, such that

$$\tau = c_d \rho U_a^2, \tag{1.2}$$

we obtain

$$\tau_w/\tau = (0{\cdot}68 \times 10^{-6}/c_d)(g\nu_w)^{-\frac{1}{3}} U_a. \tag{1.3}$$

Introducing the empirical result (Sheppard 1958)

$$c_d = 8 \times 10^{-5} + 1{\cdot}14 \times 10^{-6} U_a \tag{1.4}$$

and substituting $g = 10^3$ and $\nu_w = 10^{-2}$, all in cgs units, we obtain

$$\tau_w/\tau = 0{\cdot}28(1 + 70U_a^{-1})^{-1}, \tag{1.5}$$

where U_a is in cm/sec (20 knots = 10^3 cm/sec). Stewart's estimate was $\tau_w/\tau = 0{\cdot}2$. [The referee has pointed out that (1.1) actually expresses the difference in stress on a surface with and without waves and therefore gives the sum of τ_w (as the symbol is used elsewhere in this paper) and any increment of skin friction associated with the change in roughness. Granting this objection, we infer that (1.5) may overestimate the shear stress associated with the momentum transfer to the surface waves.]

We also note that the mean-square slope of the surface is given by the empirical expression (Cox & Munk 1954)

$$\overline{s^2} = 1{\cdot}1 \times 10^{-4}(g\nu_w)^{-\frac{1}{3}} U_a, \tag{1.6}$$

the substitution of which in (1.3) yields

$$\tau_w/\tau = (6{\cdot}2 \times 10^{-3}/c_d)\overline{s^2}. \tag{1.7}$$

This suggests that the shorter waves, which are more important for the power spectrum of the slope, relative to that for the energy, may be especially significant in the present context (Stewart's estimate was based on the observationally predominant long waves).

We emphasize that the empirical results (1.1) and (1.4), on which the estimate (1.5) is based, do not enter the subsequent analysis; on the other hand, we shall invoke the empirical result (1.6) in the final step of estimating the reduction of profile curvature.

2. Momentum transfer

We shall assume that a statistically steady state has been achieved (such that the surface-wave displacement can be described by a stationary random function) and shall consider that portion of the turbulent boundary layer in which the *total* shear stress can be regarded as constant and the viscous shear stress neglected. We then can write

$$\tau_e(z) + \tau_w(z) = \tau \equiv \rho U_*^2, \tag{2.1}$$

where τ_e denotes the turbulent Reynolds stress and τ_w the Reynolds stress associated with the momentum transfer to the surface waves. Invoking well-known similarity arguments,† we shall assume that

$$\tau_e(z) = \rho[\kappa z U'(z)]^2, \tag{2.2}$$

where $U'(z)$ is the mean wind shear and κ is von Kármán's constant.

We shall proceed on the assumption that the surface-wave displacement is sufficiently small to permit linear superposition over a two-dimensional spectrum. We then can represent the displacement by the stochastic integral

$$\zeta(\mathbf{x}, t) = \mathscr{R}\iint e^{i(\mathbf{k}\cdot\mathbf{x}-ct)}\,dA(\mathbf{k}), \tag{2.3}$$

where $\mathbf{x} = (x, y)$ is a two-dimensional vector, and $\mathbf{k} = (k\cos\theta, k\sin\theta)$ is a vector wave-number. We shall refer the power spectrum of ζ, say Z, to c and θ, such that‡

$$Z(c,\theta) = \tfrac{1}{2}\overline{dA\,dA^*/dc\,d\theta}, \quad \overline{\zeta^2} = \int_0^\infty\int_0^\infty Z(c,\theta)\,dc\,d\theta, \tag{2.4a, b}$$

where dA^* is the complex conjugate of dA, and the overbar implies a mean value.

The Reynolds stress for a simple-harmonic surface wave is given by (Miles 1957, 1960)

$$\tau_w = \rho U_1^2 \beta k^2 \cos^2\theta\,\overline{\zeta^2} \quad (z < z_c) \tag{2.5a}$$

$$= 0 \quad (z > z_c), \tag{2.5b}$$

where

$$U(z_c)\cos\theta = c, \tag{2.6}$$

β is an energy-transfer coefficient, U_1 is a reference velocity, and $U(z)$ is assumed to be monotonic. In general, β is a function of the single parameter kz_c, but the supposition of the logarithmic profile

$$U(z) = (U_*/\kappa)\log(z/z_0) \tag{2.7}$$

renders it convenient to write

$$\beta = \beta(c/U_1\cos\theta, \Omega), \quad \Omega = gz_0/U_1^2 \tag{2.8a, b}$$

and to choose

$$U_1 = U_*/\kappa. \tag{2.9}$$

Now let us suppose that ζ is defined by (2.3) and (2.4). We then can generalize (2.5) to obtain (cf. Miles 1957, equation (8.3))

$$\tau_w(z) = \rho U_1^2 \int_0^{2\pi} \cos^2\theta\,d\theta \int_{U(z)\cos\theta}^\infty \beta k^2 Z(c,\theta)\,dc. \tag{2.10}$$

Introducing the slope spectrum

$$\Sigma(c,\theta) = k^2 Z(c,\theta), \tag{2.11}$$

we can rewrite (2.10) in the more convenient form

$$\tau_w(z) = \rho U_*^2 f(U), \tag{2.12}$$

where

$$f(U) = \left(\frac{U_1}{U_*}\right)^2 \int_0^{2\pi} \cos^2\theta\,d\theta \int_{U\cos\theta}^\infty \beta\left(\frac{c}{U_1\cos\theta}, \Omega\right)\Sigma(c,\theta)\,dc. \tag{2.13}$$

† The implicit assumption that z is the only relevant scale length for τ_e must be valid as $\tau_w/\tau_e \to 0$ and is consistent with the subsequent assumption of small surface-wave displacement.

‡ Frequency may be preferable to wave speed for observational purposes; thus, Longuet-Higgins (1962) introduces the directional spectrum $F(\sigma, \theta)$, such that $F\,d\sigma = Z\,dc$, where σ denotes angular frequency.

Substituting (2.2) and (2.12) into (2.1), we obtain

$$U'(z) = (U_*/\kappa z)\,[1-f(U)]^{\frac{1}{2}} \qquad (2.14a)$$

$$\doteqdot (U_*/\kappa z)\,[1-\tfrac{1}{2}f(U)]. \qquad (2.14b)$$

As it stands, (2.14*a*) is an implicit differential equation for $U(z)$ that yields the logarithmic profile of (2.7) in the absence of surface waves ($f \equiv 0$). Invoking the assumption of small disturbances, already implicit in (2.10), we can calculate β and f on the basis of the profile (2.7) and then calculate the next approximation to the profile by integrating (2.14*b*). Higher approximations could be obtained from (2.14*a*) by iteration, but they would be inconsistent both with the linearization of the equations of motion in the calculation of β and with linear superposition over the surface-wave spectrum.

3. Reduction in curvature

Let

$$G(z) = -zU''(z)/U'(z) \qquad (3.1)$$

be a measure of profile curvature; $G \equiv 1$ for the logarithmic profile of (2.7). Differentiating (2.14), we obtain

$$G(z) = 1+\tfrac{1}{2}(U_*/\kappa)f'(U)\,[1-f(U)]^{-\frac{1}{2}} \qquad (3.2a)$$

$$\doteqdot 1+\tfrac{1}{2}(U_*/\kappa)f'(U). \qquad (3.2b)$$

Differentiating (2.13) and invoking (2.9), we obtain the first approximation

$$f'(U) \doteqdot -\kappa^{-2}\beta\left(\frac{U}{U_1},\Omega\right)\int_{-\pi}^{\pi}\Sigma(U\cos\theta,\theta)\cos^3\theta\,d\theta. \qquad (3.3)$$

The energy-transfer coefficient β is proportional to the ratio of profile curvature to slope in the critical layer, and hence to $G(z_c)$, but otherwise it is relatively insensitive to the shape of the velocity profile. Accordingly, we can estimate the non-linear reduction of the energy transfer to a given part of the spectrum, in consequence of the profile change induced by the entire spectrum (there may be other non-linear modifications of β_0), according to

$$\beta = \beta_0\left(\frac{c}{U_1\cos\theta},\Omega\right)G(z_c), \qquad (3.4)$$

where β_0 is based on (2.7), and $G(z)$ is given by (3.2*b*) after approximating β by β_0 in (3.3). Combining (3.2*b*)–(3.4), we can place the result in the form

$$\frac{1}{\beta}-\frac{1}{\beta_0} = \frac{1}{2}\frac{U_1}{\kappa^2}\int_{-\pi}^{\pi}\Sigma\left(c\frac{\cos\phi}{\cos\theta},\phi\right)\cos^3\phi\,d\phi. \qquad (3.5)$$

It will suffice for a rough estimate of G to assume the Neumann spectrum (see *Ocean Wave Spectra* 1963, especially Longuet-Higgins, Cartwright & Smith, for more accurate approximations)

$$\Sigma(c,\theta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\overline{s^2}}{U_a\theta_0}\exp\left[-2\left(\frac{c}{U_a}\right)^2\right]\begin{Bmatrix}1\\0\end{Bmatrix}, \quad |\theta| \lessgtr \theta_0, \qquad (3.6)$$

where $2\theta_0$ is the angular width of the spectrum.

Substituting (3.6) into (3.3), we obtain

$$f'(U) = -\left(\frac{8}{\pi}\right)^{\frac{1}{2}} \frac{\overline{s^2}}{\kappa^2 U_a} \beta\left(\frac{U}{U_a}, \Omega\right) \Phi\left(\frac{U}{U_a}, \theta_0\right), \tag{3.7}$$

where
$$\Phi(u, \theta_0) = \frac{1}{2\theta_0}\int_{-\theta_0}^{\theta_0} \exp(-2u^2\cos^2\theta)\cos^3\theta \, d\theta. \tag{3.8}$$

This last integral can be expressed in terms of tabulated functions, but we shall rest content with the approximation

$$\Phi(u, \theta_0) = \exp(-2u^2\cos^2\theta_0)\,[1 - \tfrac{1}{3}\sin^2\theta_0]\,(\sin\theta_0/\theta_0)\,[1 + O(u^2)]. \tag{3.9}$$

We add that (3.9) gives an upper bound to Φ and has a maximum value of unity (at $u = \theta_0 = 0$).

Substituting the mean-square slope into (3.7) from (1.6), setting $\kappa = 0{\cdot}4$, $g = 10^3$ and $\nu_w = 10^{-2}$, and then substituting the result into (3.2*b*), we obtain

$$G(z) = 1 - 6{\cdot}3 \times 10^{-4} U_* \beta(U/U_1, \Omega)\,\Phi(U/U_a, \theta_0), \tag{3.10}$$

where U_* is in cm/sec. The factor $\beta\Phi$ cannot exceed a value of roughly 2, corresponding to a minimum value of $1 - 10^{-3}U_*$ for G; e.g. $G > 0{\cdot}94$ for $U_a = 20$ knots.

4. Conclusion

We emphasize that the analysis of (2.1)–(3.5) rests on a number of rational, but still not firmly established, assumptions. Moreover, our numerical estimate of G rests on the additional (and less rational) assumptions of the Neumann spectrum (3.6) and of the empirical expression (1.6) for the mean-square slope. Thus, the assumption of some other spectrum, in place of (3.6), could easily alter Φ and hence $1 - G$ by a factor of 2 (e.g. the assumption $\Sigma \equiv 0$ for $c > \frac{1}{2}U_a$ would require the right-hand side of (3.6) to be increased by the factor 1·47 for $c > \frac{1}{2}U_a$, with a corresponding increase of Φ).

With these reservations, we conclude that the reductions of profile curvature and slope owing to momentum transfer from a turbulent wind to surface waves are small for typical wind speeds.

This research has been partially supported by the Office of Naval Research under Contract Nonr-4266(00).

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